MATH 4140/7140 Homework 5 Due 7/21/2016

Do all problems on different paper. I will not grade anything written on this page. Show your work. Answers without work will be given little or no credit.

1. (a) Show that for any $\mathbf{v}, \mathbf{w}$ in an inner product space V, the following identity holds:

$$||\,\mathbf{v}+\mathbf{w}\,||^2 + ||\,\mathbf{v}-\mathbf{w}\,||^2 = 2||\,\mathbf{v}\,||^2 + 2||\,\mathbf{w}\,||^2$$

(We know that an inner product space is always a normed linear space as a norm can be defined from the inner product. However, not every normed linear space is an inner product space, meaning the norm may not arise from an inner product. In fact, a normed linear space is an inner product space if and only if the norm satisfies the above equality for every $\mathbf{v}, \mathbf{w} \in V$.)

   (b) Show that the infinity norm $||\cdot||_\infty$ on $\mathbb{R}^2$ defined by $||(x_1, x_2)^T||_\infty = \max\{|x_1|, |x_2|\}$ is not derived from an inner product.

2. Consider the vector space $C[-1, 1]$ with the standard inner product, that is,

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Find an orthonormal basis for the subspace spanned by $1, x, x^2$.

3. Consider the following orthonormal basis for $\mathbb{R}^3$ (you do not have to show it is an orthonormal basis):

$$\mathbf{u}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T, \ \mathbf{u}_2 = \left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)^T, \ \mathbf{u}_3 = \left(\frac{4}{\sqrt{42}}, \frac{-5}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right)^T.$$

Use the inner product to write $\mathbf{v} = (-1, 1, 2)^T$ as a linear combination of these basis vectors. Then find the length of $\mathbf{v}$ using Parseval's formula.

4. Recall that if V is a vector space and $\varphi : V \to V$ is a linear operator, then we say that a <u>nonzero</u> vector $\mathbf{v} \in V$ is an *eigenvector* of V if $\varphi(\mathbf{v}) = \lambda\,\mathbf{v}$ for some scalar λ, when this happens we say that λ is an *eigenvalue* of φ.

   (a) Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) Let $V = \mathbb{R}^\infty = \{(x_1, x_2, ..., x_n, ...)^T \mid x_i \in \mathbb{R}\}$. Notice that $\mathbb{R}^\infty$ is a vector space with addition defined by component-wise addition and scalar multiplication defined by $\alpha \cdot (x_1, x_2, ..., x_n, ...)^T = (\alpha x_1, \alpha x_2, ..., \alpha x_n, ...)^T$. Consider the linear operator $\varphi : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $\varphi((x_1, x_2, ..., x_n, ...)^T) = (0, x_1, x_2, ..., x_n, ...)^T$. (You do not need to show that $\mathbb{R}^\infty$ is a vector space nor do you need to show φ is a linear operator). Show that φ does not have an eigenvector.

(c) Consider the differential operator $D : C^\infty(-\infty, \infty) \to C^\infty(-\infty, \infty)$ defined by $D(f(x)) = f'(x)$. Show that every $\lambda \in \mathbb{R}$ is an eigenvalue of D. (Hint: Consider a function that is closely related to its derivative.)